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On Differentiability of the Operator of Best L_1-Approximation

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1

Let X be a normed linear space and consider an n-dimensional subspace $U_n \subset X$. Assume that U_n is a *unicity subspace* of X, that is, each $f \in X$ possesses a unique best approximant $q \in U_n$ for which $\|f-q\| = \inf\{\|f-p\|: p \in U_n\}$. In this case we can consider the *best approximation operator* $\mathscr{P}: X \to U_n$ mapping each $f \in X$ into its best approximant in U_n. This operator being bounded and continuous, is at the same time in general nonlinear. (The linearity of $\mathscr{P}$ is essentially characteristic only for Hilbert spaces, see [8, p. 249].) This leads to the natural desire to approximate $\mathscr{P}$ in a neighborhood of $f \in X$ by a linear operator or, in other words, to the question of *differentiability of the best approximation operator*.

The operator $\mathscr{P}$ has at $f \in X$ a one-sided derivative, denoted by $D_f\mathscr{P}$: $X \to U_n$, if, for each $g \in X$, the limit

$$\lim_{t \to +0} \frac{\mathscr{P}(f+tg) - \mathscr{P}(f)}{t} = D_f\mathscr{P}(g)$$

exists. In case $D_f\mathscr{P}(g) = -D_f\mathscr{P}(-g)$, $g \in X$, we say that $\mathscr{P}$ is differentiable at f. If, in addition, the derivative $D_f\mathscr{P}$ is a linear operator of direction g, then $\mathscr{P}$ is called *Gatoux differentiable* at $f \in X$.

When $X = L_p$, $2 < p < \infty$, the differentiability of the operator $\mathscr{P}$ was studied by Kripke and Holmes [4]. It was shown in [4] that in L_p, $2 < p < \infty$, the operator $\mathscr{P}$ is differentiable at each $f \in L_p$ but is not, in general, Gatoux differentiable. In [5] the differentiability of $\mathscr{P}$ was investigated for the case $X = C[a, b]$. We proved in [5] that the operator of best Chebyshev approximation has a one-sided derivative at each $f \in C[a, b]$ and characterized those functions in $C[a, b]$ at which $\mathscr{P}$ is Gatoux differen-

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tiable. In the present paper we shall study the differentiability of the operator of best L_1-approximation. The Gatoux differentiability of $\mathscr{P}$ will be verified for the important class of the so-called generalized convex functions. We also give some applications for polynomials and spline functions.

2

It is well known that the space L_1 does not possess unicity subspaces. Therefore the L_1-approximation problem will be studied in the space $X = C_1$ consisting of all real-valued continuous functions on $[-1, 1]$ and endowed with the L_1-norm on $[-1, 1]$. We consider an n-dimensional unicity subspace U_n in C_1 with basis $\{u_i\}_{i=1}^n$ and denote by $\mathscr{P}: C_1 \to U_n$ the operator of best L_1-approximation. Since all known unicity subspaces of C_1 (Haar subspaces, different families of spline functions) satisfy the weak Chebyshev property it is natural to assume that U_n is also a weak Chebyshev space (for the definition see, e.g., [9]). In what follows the unicity subspaces of C_1 which are also weak Chebyshev spaces will be called *UW-spaces*. We shall also impose on the subspace U_n some nondegeneracy conditions. As usual a point $\xi \in [-1, 1]$ is called essential relative to U_n if not all elements of U_n vanish at ξ. We shall say that U_n, $n \geqslant 2$, is *nondegenerate* provided (a) all points in $[-1, 1]$ are essential relative to U_n; (b) for each $\xi \in (-1, 1)$ we can find $g \in U_n$ such that $g(\xi) = 0$ and

$$\liminf_{x \to \xi} \left| \frac{g(x)}{\xi - x} \right| > 0. \tag{1}$$

(In case the elements of U_n are differentiable (1) is equivalent to $g'(\xi) \neq 0$.)

Assume now that U_n is a UW-space. Then by a result of Sommer [10] (see also Micchelli [7]) there exists a unique set $\tau_1, \tau_2, ..., \tau_n$ of canonical points: $-1 = \tau_0 < \tau_1 < \cdots < \tau_n < \tau_{n+1} = 1$ such that

$$\sum_{i=0}^{n} (-1)^i \int_{\tau_i}^{\tau_{i+1}} q(x)\, dx = 0, \qquad q \in U_n. \tag{2}$$

Moreover, if $u_1, ..., u_n$ is a basis in U_n then

$$\det \begin{pmatrix} u_1, ..., u_n \\ \tau_1, ..., \tau_n \end{pmatrix} \neq 0,$$

that is, we may interpolate at the nodes $\tau = \{\tau_i\}_{i=1}^n$. Let us denote by $\mathscr{L}_\tau$: $C_1 \to U_n$ the corresponding interpolation operator, i.e., $\mathscr{L}_\tau(f, \tau_i) = f(\tau_i)$, $1 \leqslant i \leqslant n, f \in C_1$. For a weak Chebyshev space U_n with basis $\{u_i\}_{i=1}^n$ denote by $K(U_n)$ its convexity cone or the cone of generalized convex functions consisting of those continuous functions f for which the determinant

$$\det\begin{pmatrix} u_1,\ldots,u_n,f \\ x_1,\ldots,x_n,x_{n+1}\end{pmatrix}$$

preserves its sign (is nonnegative or nonpositive) for any $-1 < x_1 < \cdots < x_{n+1} < 1$. The L_1-approximation problem for generalized convex functions was studied by Micchelli [7]. In particular it was shown in [7] that $\mathscr{P}(f) = \mathscr{L}_\tau(f)$ provided $f \in K(U_n)$.

In what follows Lip 1 denotes the set of point-wise Lipschitz continuous functions on $(-1, 1)$, i.e., Lip $1 = \{f$: for each $x \in (-1, 1)$ we can find a constant M_x such that $|f(x) - f(y)| \leqslant M_x |x-y|$ for any $y \in (-1, 1)\}$.

Our main result is the following

THEOREM 1. *Let U_n, $n \geqslant 2$, be a nondegenerate UW-space such that $U_n \subset$ Lip 1, and consider an arbitrary $f \in K(U_n) \cap$ Lip 1. If f is not identically equal to $\mathscr{L}_\tau(f)$ on some nondegenerate interval, then the operator $\mathscr{P}$ of best L_1-approximation is Gatoux differentiable at f and $D_f\mathscr{P} = \mathscr{L}_\tau$.*

3

The proof of Theorem 1 will be based on several lemmas. We shall need the following result on zeros of elements of weak Chebyshev spaces (see [9, p. 42]).

LEMMA 1. *Let U_n be an n-dimensional weak Chebyshev space and assume that all points of $[-1, 1]$ are essential relative to U_n. Then each $g \in U_n \backslash \{0\}$ has at most n separated zeros. Moreover if either $g(1) \neq 0$ or $g(-1) \neq 0$ then the number of separated zeros of g does not exceed $n-1$.*

The next lemma will play a crucial part in proof of Theorem 1. It shows that under the hypothesis of Theorem 1 the operator $\mathscr{P}$ satisfies some kind of Lip 1 property. In what follows $\|g\|_\infty = \sup_{x \in [-1,1]} |g(x)|$ and $\|g\|_1 = \int_{-1}^{1} |g(x)|\, dx$ denote the supremum and L_1-norm of g, respectively.

LEMMA 2. *Let U_n, $n \geqslant 2$, be a nondegenerate UW-space, $U_n \subset$ Lip 1 and consider an arbitrary $f \in K(U_n) \cap$ Lip 1 such that $f - \mathscr{L}_\tau(f)$ is not identically zero on some nondegenerate interval. Then for any bounded measurable function g and any best L_1-approximant $\tilde{g}$ to g from U_n we have*

$$\|\mathscr{P}(f) - \tilde{g}\|_\infty \leqslant M(f)\, \|f - g\|_\infty,$$

where the constant $M(f) > 0$ depends only on f and U_n.

Proof of Lemma 2. Since $f \in K(U_n)$ it follows that $\mathscr{P}(f) = \mathscr{L}_\tau(f)$. We may assume that $\mathscr{L}_\tau(f) \equiv 0$, i.e.,

$$f(x) = \frac{\det \begin{pmatrix} u_1,..., u_n, f \\ \tau_1,..., \tau_n, x \end{pmatrix}}{\det \begin{pmatrix} u_1,..., u_n \\ \tau_1,..., \tau_n \end{pmatrix}},$$

where as above $\{u_i\}_{i=1}^n$ and $\{\tau_i\}_{i=1}^n$ are the basis and canonical set of U_n, respectively. Hence

$$\gamma(-1)^i f(x) \geqslant 0, \qquad x \in [\tau_i, \tau_{i+1}], \tag{3}$$

where $|\gamma| = 1$ and $0 \leqslant i \leqslant n$. Furthermore, by our assumption f does not vanish on a nondegenerate interval, i.e., all its zeros are separated. On the other hand f belongs to the $(n+1)$-dimensional weak Chebyshev space spanned by $u_1,..., u_n$ and f. Thus applying Lemma 1 we may conclude that f vanishes only at τ_i, $1 \leqslant i \leqslant n$. Moreover we state that for any $1 \leqslant i \leqslant n$,

$$\liminf_{x \to \tau_i} \left| \frac{f(x)}{x - \tau_i} \right| > 0. \tag{4}$$

Assume the contrary, that is, for some $1 \leqslant j \leqslant n$ and $x_k \to \tau_j$ we have

$$\lim_{x_k \to \tau_j} \frac{|f(x_k)|}{|x_k - \tau_j|} = 0. \tag{5}$$

We may assume without loss of generality that $x_k \to +\tau_j$. Nondegeneracy of U_n implies that there exists $p \in U_n$, $\|p\|_\infty = 1$ such that $p(\tau_j) = 0$, and $\gamma(-1)^j p(x) \geqslant \eta(x - \tau_j)$ for some $\eta > 0$ and any $\tau_j \leqslant x \leqslant \tau_j + h$, $h > 0$ (see (1)). Now let ε be a positive number such that

$$\varepsilon < \min_{0 \leqslant i \leqslant n} \max_{x \in [\tau_i, \tau_{i+1}]} |f(x)| \tag{6}$$

and

$$\varepsilon < \min\{|f(-1)|, |f(1)|\}, \tag{7}$$

and consider the function $f_1 = f - \varepsilon p$. It follows from (3) and (6) that for some $\xi_i \in (\tau_i, \tau_{i+1})$,

$$\gamma(-1)^i f(\xi_i) > \varepsilon, \qquad 0 \leqslant i \leqslant n.$$

Therefore $\gamma(-1)^i f_1(\xi_i) > \varepsilon - \varepsilon\gamma(-1)^i p(\xi_i) \geqslant 0$, i.e., f_1 has at $(\xi_0, \xi_{j-1}) \cup (\xi_j, \xi_n)$, $n-1$ separated zeros. Furthermore $f_1(\tau_j) = 0$ and by (3) and (5)

$$\gamma(-1)^j f_1(x_k) = |f(x_k)| - \varepsilon\gamma(-1)^j p(x_k)$$
$$\leqslant (x_k - \tau_j) \left\{ \frac{|f(x_k)|}{(x_k - \tau_j)} - \varepsilon\eta \right\} < 0$$

for k large enough. Since $\gamma(-1)^j f_1(\xi_j) > 0$ and $\tau_j < x_k < \xi_j$ for k sufficiently large we may conclude that (ξ_{j-1}, ξ_j) contains 2 additional separated zeros of f_1. Hence f_1 has at least $n+1$ separated zeros. Furthermore since f_1 is an element of an $(n+1)$-dimensional weak Chebyshev space it follows from Lemma 1 that $f_1(1) = f_1(-1) = 0$. But this evidently contradicts (7). By this contradiction we obtain that (4) holds for each $1 \leqslant i \leqslant n$.

Consider the set

$$A(t) = \{x \in [-1, 1]: |f(x)| \leqslant t\}, \qquad 0 < t < \|f\|_\infty. \tag{8}$$

Since f vanishes only at τ_i, $1 \leqslant i \leqslant n$, and by (4) it tends to zero as $x \to \tau_i$ at most, linearly it follows that

$$\mu(A(t)) \leqslant c_1 t, \tag{9}$$

where $\mu(\cdots)$ denotes the Lebesgue measure. (Here and in the remaining part of proof of Theorem 1 we denote by c_1, c_2,..., positive constants depending only on f and U_n.)

Now let g be a bounded measurable function, $\tilde{g} \in U_n$ any of its best L_1-approximants in U_n. Set $\|f - g\|_\infty = \varepsilon$, $\varepsilon \geqslant 0$. We may assume without loss of generality that $\varepsilon < \|f\|_\infty$ since otherwise

$$\begin{aligned} \|\tilde{g}\|_1 &\leqslant \|f\|_1 + \|f - g\|_1 + \|g - \tilde{g}\|_1 \\ &\leqslant \|f\|_1 + \|f - g\|_1 + \|g\|_1 \leqslant 2\|f\|_1 + 2\|f - g\|_1 \\ &\leqslant 4\|f\|_\infty + 4\|f - g\|_\infty \leqslant 8\|f - g\|_\infty, \end{aligned}$$

and the statement of Lemma 2 follows by equivalence of norms in finite-dimensional spaces. For $\varepsilon < \|f\|_\infty$ we consider the function

$$\begin{aligned} f_\varepsilon(x) &= f(x), \qquad x \in A(\varepsilon), \\ &= g(x), \qquad x \in [-1, 1] \setminus A(\varepsilon). \end{aligned} \tag{10}$$

Using that $\|f - g\|_\infty = \varepsilon$ we obtain by (3) that

$$\operatorname{sign} f_\varepsilon(x) = \operatorname{sign} f(x) = \gamma(-1)^i, \qquad x \in (\tau_i, \tau_{i+1}), \quad 0 \leqslant i \leqslant n.$$

Hence (2) yields that

$$\|f_\varepsilon - \tilde{g}\|_1 - \|f_\varepsilon\|_1 = \int_{-1}^{1} (f_\varepsilon - \tilde{g})\{\operatorname{sign}(f_\varepsilon - \tilde{g}) - \operatorname{sign} f_\varepsilon\}$$

$$= 2 \int_{B(f_\varepsilon, \tilde{g})} |f_\varepsilon - \tilde{g}|, \tag{11}$$

where $B(f_\varepsilon, \tilde{g}) = \{x \in [-1, 1]: 0 < f_\varepsilon(x) < \tilde{g}(x) \text{ or } \tilde{g}(x) < f_\varepsilon(x) < 0\}$.

Let us prove that for any $x \in [-1, 1]$ and $1 \leqslant i \leqslant n$ we have with a suitable $c_2 > 0$,

$$|f_\varepsilon(x)| \leqslant c_2 |x - \tau_i|. \tag{12}$$

If $x \in A(\varepsilon)$ then $f_\varepsilon(x) = f(x)$ and (12) follows by $f(\tau_i) = 0$ and the Lip 1 property of f at τ_i. On the other hand, $\varepsilon \leqslant |f(x)| \leqslant c_3 |x - \tau_i|$ when $x \in [-1, 1] \backslash A(\varepsilon)$. Hence by (10),

$$|f_\varepsilon(x)| = |g(x)| \leqslant |f(x)| + \varepsilon \leqslant 2c_3 |x - \tau_i|,$$

i.e., (12) holds for every $x \in [-1, 1]$.

Furthermore, since $U_n \subset \text{Lip } 1$, it easily follows that for any $p \in U_n$ and $1 \leqslant i \leqslant n$,

$$|p(x) - p(\tau_i)| \leqslant c_4 \|p\|_\infty |x - \tau_i|. \tag{13}$$

Moreover, using that $\{\tau_i\}_{i=1}^n$ is an interpolation set for U_n, we may conclude that

$$\|p\|_\infty \leqslant c_5 \max_{1 \leqslant i \leqslant n} |p(\tau_i)|, \qquad p \in U_n. \tag{14}$$

Applying (13) and (14) for $\tilde{g} \in U_n$ we obtain that for some $1 \leqslant j \leqslant n$ and any $x \in [-1/2c_4c_5 + \tau_j, \tau_j + 1/2c_4c_5]$,

$$|\tilde{g}(x)| \geqslant |\tilde{g}(\tau_j)| - |\tilde{g}(x) - \tilde{g}(\tau_j)| \geqslant \frac{\|\tilde{g}\|_\infty}{c_5} - c_4 \|\tilde{g}\|_\infty |x - \tau_j|$$

$$\geqslant \frac{\|\tilde{g}\|_\infty}{c_5} - \frac{\|\tilde{g}\|_\infty}{2c_5} = \frac{\|\tilde{g}\|_\infty}{2c_5}. \tag{15}$$

We may assume without loss of generality that $f_\varepsilon > 0$ on (τ_j, τ_{j+1}) and $\tilde{g} > 0$ on $[-1/2c_4c_5 + \tau_j, \tau_j + 1/2c_4c_5]$. Set $\tau_j^* = \min\{\tau_{j+1}, \tau_j + \|\tilde{g}\|_\infty / 4c_2c_5\}$. Then (12) implies that for $x \in (\tau_j, \tau_j^*)$,

$$0 < f_\varepsilon(x) \leqslant \frac{\|\tilde{g}\|_\infty}{4c_5}. \tag{16}$$

Hence by (15), $(\tau_j, \tau_j^*) \subset B(f_\varepsilon, \tilde{g})$. Thus it follows from (11), (15), and (16),

$$\|f_\varepsilon - \tilde{g}\|_1 - \|f_\varepsilon\|_1 \geqslant \int_{\tau_j}^{\tau_j^*} |f_\varepsilon - \tilde{g}| \geqslant (\tau_j^* - \tau_j) \frac{\|\tilde{g}\|_\infty}{4c_5}$$
$$\geqslant c_6 \min\{\|\tilde{g}\|_\infty, \|\tilde{g}\|_\infty^2\}. \tag{17}$$

On the other hand, using that $\tilde{g}$ is a best L_1-approximant of g, we can obtain the following upper estimate

$$\|f_\varepsilon - \tilde{g}\|_1 - \|f_\varepsilon\|_1 \leqslant \|f_\varepsilon - g\|_1 + \|g - \tilde{g}\|_1 - \|f_\varepsilon\|_1$$
$$\leqslant \|f_\varepsilon - g\|_1 + \|g\|_1 - \|f_\varepsilon\|_1 \leqslant 2\|f_\varepsilon - g\|_1.$$

Hence (10) and (9) yield

$$\|f_\varepsilon - \tilde{g}\|_1 - \|f_\varepsilon\|_1 \leqslant 2\|f_\varepsilon - g\|_1 = 2\int_{A(\varepsilon)} |f - g| \leqslant 2\varepsilon\mu(A(\varepsilon)) \leqslant 2c_1\varepsilon^2.$$

Combining this with (17) and taking into account that $\varepsilon < \|f\|_\infty$, we finally obtain

$$\|\tilde{g}\|_\infty \leqslant c_7\varepsilon = c_7\|f - g\|_\infty.$$

The proof of the lemma is completed.

Remark 1. The main point in Lemma 2 is that we estimate the distance between best L_1-approximants of functions f and g while the deviation of g from f is measured in supremum norm. This leads to a Lip 1 type property. On the other hand, if the distance between g and f is measured in L_1-norm, then the operator $\mathscr{P}$ satisfies only the Lip $\frac{1}{2}$ condition, the proof being similar to [6, p. 341].

For functions $f \in K(U_n)$ we have the nice relation $\mathscr{P}(f) = \mathscr{L}_\tau(f)$. Our following lemma shows that this relation almost holds for nearby functions. By $\omega(g, h) = \max\{|g(x_1) - g(x_2)|: x_1, x_2 \in [-1, 1], |x_1 - x_2| \leqslant h\}$ we denote the modulus of continuity of a continuous function g.

LEMMA 3. *Let U_n, $n \geqslant 2$, be a nondegenerate UW-space, $U_n \subset$ Lip 1 and consider an arbitrary $f \in K(U_n) \cap$ Lip 1 such that $f - \mathscr{L}_\tau(f)$ is not identically zero on a nondegenerate interval. Then for any continuous function g with $\|g\|_\infty = 1$ and $0 < t < t_0(f)$ we have*

$$\|\mathscr{P}(f + tg) - \mathscr{L}_\tau(f + tg)\|_\infty \leqslant M_1(f)\, t\omega(g - \mathscr{L}_\tau(g), t),$$

where $t_0(f)$ and $M_1(f)$ are positive constants independent of g.

Proof of Lemma 3. Set $f^* = f - \mathscr{L}_\tau(f)$, $g^* = g - \mathscr{L}_\tau(g)$. Evidently $\mathscr{P}(f^*) \equiv 0$. Hence by Lemma 2,

$$\|\mathscr{P}(f^* + tg^*)\|_\infty \leqslant M(f)t \, \|g^*\|_\infty. \tag{18}$$

For $\varepsilon_t = (M(f)+1) \, \|g^*\|_\infty \, t$ we consider the function

$$\begin{aligned} f_t(x) &= f^*(x) + tg^*(x), \qquad && x \in A(\varepsilon_t), \\ &= f^*(x), && x \in [-1, 1] \backslash A(\varepsilon_t), \end{aligned} \tag{19}$$

where $A(h) = \{x \in [-1, 1] : |f^*(x)| \leqslant h\}$.

Let us verify that for any $x \in [-1, 1]$,

$$\operatorname{sign}(f_t - \mathscr{P}(f^* + tg^*)) = \operatorname{sign}(f^* + tg^* - \mathscr{P}(f^* + tg^*)). \tag{20}$$

For $x \in A(\varepsilon_t)$ this holds automatically. Furthermore, if $x \in [-1, 1] \backslash A(\varepsilon_t)$, then $f_t = f^*$ and by (18),

$$|tg^* - \mathscr{P}(f^* + tg^*)| \leqslant \varepsilon_t < |f^*|.$$

Hence (20) is true for any $x \in [-1, 1]$.

It is known (see [8, p. 46]) that since $\mathscr{P}(f^* + tg^*)$ is the best L_1-approximant of $f^* + tg^*$ we have

$$\left| \int_{-1}^{1} p \operatorname{sign}(f^* + tg^* - \mathscr{P}(f^* + tg^*)) \right| \leqslant \int_Z |p|, \qquad p \in U_n,$$

where $Z = \{x \in [-1, 1] : f^*(x) + tg^*(x) = \mathscr{P}(f^* + tg^*, x)\}$, $Z \subseteq A(\varepsilon_t)$. Hence by (20), for any $p \in U_n$,

$$\begin{aligned} &\|f_t - \mathscr{P}(f^* + tg^*)\|_1 \\ &\quad = \int_{-1}^{1} (f_t - \mathscr{P}(f^* + tg^*) \operatorname{sign}(f^* + tg^* - \mathscr{P}(f^* + tg^*)) \\ &\quad \leqslant \int_{[-1,1] \backslash Z} |f_t - p| \\ &\qquad + \left| \int_{-1}^{1} (p - \mathscr{P}(f^* + tg^*)) \operatorname{sign}(f^* + tg^* - \mathscr{P}(f^* + tg^*)) \right| \\ &\quad \leqslant \int_{[-1,1] \backslash Z} |f_t - p| + \int_Z |p - \mathscr{P}(f^* + tg^*)| = \|f_t - p\|_1, \end{aligned}$$

i.e., $\mathscr{P}(f^* + tg^*)$ is a best L_1-approximant of f_t in U_n. Then again applying Lemma 2 for functions f and $f_t + \mathscr{L}_\tau(f)$ we obtain by (19),

$$\|\mathscr{P}(f^* + tg^*)\|_\infty \leqslant M(f) \, \|f^* - f_t\|_\infty \leqslant M(f)t \max_{x \in A(\varepsilon_t)} |g^*(x)|. \tag{21}$$

Using once more relations (4) (with $f = f^*$), we may conclude that for any $0 < t < t_1$,

$$A(t) \subset \bigcup_{i=1}^{n} (-bt + \tau_i, \tau_i + bt), \tag{22}$$

where t_1 and $b > 0$ depend only on f. Furthermore, since $\mathscr{L}_\tau$ is a bounded linear operator

$$\|g^*\|_\infty \leqslant 1 + \|L_\tau(g)\|_\infty \leqslant R, \tag{23}$$

the upper bound being independent of g. Hence by (22), for any $0 < t < t_0 = t_1/R(M(f)+1)$,

$$A(\varepsilon_t) \subset \bigcup_{i=1}^{n} (-b\varepsilon_t + \tau_i, \tau_i + b\varepsilon_t). \tag{24}$$

Moreover $g^*(\tau_i) = 0$, $1 \leqslant i \leqslant n$. Thus (24) and (23) imply that

$$\max_{x \in A(\varepsilon_t)} |g^*(x)| \leqslant \omega(g^*, b\varepsilon_t)$$

$$\leqslant ((M(f)+1)Rb + 1)\, \omega(g - \mathscr{L}_\tau(g), t).$$

Substituting this into (21) we obtain the estimation of Lemma 3.

Evidently Theorem 1 is a straightforward consequence of Lemma 3. In case $n = 1$ we can prove a slightly more general result than Theorem 1. We set $U_1 = \{c\phi : c \in \mathbb{R}\}$, where ϕ is a positive continuous function, $\phi(x) \geqslant \phi_0 > 0$, $x \in [-1, 1]$. Then $K(U_1)$ contains those continuous functions f for which f/ϕ is monotone. By the Jackson–Krein theorem (see [8, p. 236]) U_1 is a unicity subspace of C_1. The canonical set of U_1 consists of one point $-1 < \tau_1 < 1$ such that

$$\int_{-1}^{\tau_1} \phi - \int_{\tau_1}^{1} \phi = 0. \tag{25}$$

Theorem 2. *Let $U_1 = \{c\phi : c \in \mathbb{R}\}$, where ϕ is continuous and positive and consider a continuous function f such that f/ϕ is monotone and f is not identically equal to $\mathscr{L}_\tau(f)$ on some nondegenerate interval. Then operator $\mathscr{P}$ of best L_1-approximation is Gatoux differentiable at f and $D_f\mathscr{P} = \mathscr{L}_\tau$.*

Proof of Theorem 2. Consider an arbitrary continuous function g with $\|g\|_\infty = 1$. Without loss of generality we may assume that $f(\tau_1) = g(\tau_1) = 0$, i.e., $\mathscr{L}_\tau(f) = \mathscr{L}_\tau(g) \equiv 0$. Then in order to prove the theorem we have to show that $\mathscr{P}(f + tg)/t \to 0$ as $t \to 0$. Since f/ϕ is monotone and f does not vanish on a nondegenerate interval it follows that τ_1 is the only zero of f/ϕ and

$$A^*(t) = \left\{ x \in [-1, 1] : \left| \frac{f(x)}{\phi(x)} \right| \leqslant \frac{|t|}{\phi_0} \right\} = [\tau_1 - h_1(t), \tau_1 + h_2(t)],$$

where $h(t)=\max\{h_1(t), h_2(t)\} \to +0$ as $t \to 0$. Set $\mathscr{P}(f+tg)=c_t\phi$, $\eta(t)=\omega(g, h(t))$. We claim that for any t,

$$|c_t| \leqslant \frac{|t|}{\phi_0}\eta(t) \tag{26}$$

which immediately implies that $\mathscr{P}(f+tg)/t \to 0$ $(t \to 0)$. Assume that (26) is not true, i.e.,

$$\gamma c_t > \frac{|t|}{\phi_0}\eta(t) \qquad (|\gamma|=1). \tag{27}$$

Of course we may assume that $\gamma f<0$ on $(\tau_1, 1]$ (since τ_1 is the only zero of f and f changes its sign at τ_1). Let us verify that

$$\gamma(f+tg-c_t\phi)<0 \tag{28}$$

on $[\tau_1, 1]$. If $\gamma(f(x)+tg(x)) \leqslant 0$ then by (27),

$$\gamma(f(x)+tg(x)-c_t\phi(x)) \leqslant -\gamma c_t\phi(x) < -|t|\,\eta(t) \leqslant 0.$$

On the other hand, if $x \in [\tau_1, 1]$ and $\gamma(f(x)+tg(x))>0$ then $0 \geqslant \gamma f(x) > -\gamma t g(x) \geqslant -|t|$, i.e., $x \in A^*(t)$. Hence using that $g(\tau_1)=0$ we obtain by (27),

$$\begin{aligned}\gamma(f(x)+tg(x)-c_t\phi(x)) &\leqslant t\gamma g(x)-\gamma c_t\phi(x)\\ &< -|t|\,\eta(t)+|t|\,\omega(g, h(t))=0.\end{aligned}$$

Thus (28) holds on $[\tau_1, 1]$. This means that it holds also on $[\tau_1-\delta, 1]$ and with $\tilde{c}_t=c_t-\gamma\varepsilon$ for some $\varepsilon, \delta>0$. Hence by (25),

$$\begin{aligned}\|f+tg-\tilde{c}_t\phi\|_1-\|f+tg-c_t\phi\|_1 &\leqslant \varepsilon\int_{-1}^{\tau_1-\delta}\phi-\varepsilon\int_{\tau_1-\delta}^{1}\phi\\ &= -2\varepsilon\int_{\tau_1-\delta}^{\tau_1}\phi<0\end{aligned}$$

which is an evident contradiction since $c_t\phi$ is the best L_1-approximation to $f+tg$ from U_1. The theorem is proved.

Remark 2. Let us show that if we drop in the above theorem the condition that f is not identically equal to $\mathscr{L}_\tau(f)$ on a nondegenerate interval then differentiability may fail. Consider the case when U_1 is the space of constants and set

$$\begin{aligned} f(x) &= x, && x \in [0, 1],\\ &= 0, && x \in [-1, 0], \quad g(x)=f(-x). \end{aligned}$$

Then obviously $f \in K(U_1)$. Furthermore $\mathscr{P}(f)=0$, $\mathscr{P}(f+tg)=t/(t+1)$ if $t>0$ and $\mathscr{P}(f+tg)=0$ if $t<0$. Thus the right and left derivative of $\mathscr{P}$ at f in direction g is equal to 1 and 0, respectively. Hence $\mathscr{P}$ is not differentiable at f. The same remark holds in connection with Theorem 1.

Remark 3. A higher type of differentiability which can be imposed on operator $\mathscr{P}$ is the Fréchet derivative. It was shown in [3] that the operator of best Chebyshev approximation is not Fréchet differentiable. It can be also shown that under the hypothesis of Theorem 1 (or Theorem 2) the operator $\mathscr{P}$ of best L_1-approximation is not Fréchet differentiable.

4

Let us consider some applications of the theorems proved above. Assume that U_n is an n-dimensional ECT-space (see [9, p. 364], for the definition). By the Jackson–Krein theorem U_n is a unicity subspace of C_1 and hence it is evidently a UW-space. If $n \geqslant 2$, then by the definition of ECT-spaces U_n satisfies the smoothness and nondegeneracy conditions imposed in Theorem 1. Moreover it is known (see [2, p. 380]) that $K(U_n) \subset \text{Lip}\, 1$ if $n \geqslant 2$. On the other hand, a one-dimensional ECT-space is simply a linear span of a positive continuous function. Thus using Theorems 1 and 2 we obtain

COROLLARY 1. *Let U_n, $n \geqslant 1$, be an* ECT-*space. Then for any $f \in K(U_n)$ such that $f - \mathscr{L}_\tau(f)$ is not identically zero on a nondegenerate interval the operator $\mathscr{P}$ of best L_1-approximation is Gatoux differentiable at f and $D_f\mathscr{P} = \mathscr{L}_\tau$.*

A standard example of an ECT-space is the set P_n of algebraic polynomials of degree at most $n-1$. In the polynomial case the canonical points are the zeros of the Chebyshev polynomial of degree n of second kind. The cone $K(P_n)$ of generalized convex functions contains continuous functions f whose nth order divided difference

$$d_n(f, x_1, ..., x_{n+1}) = \sum_{i=1}^{n+1} \frac{f(x_i)}{\omega'(x_i)} \qquad \left(\omega(x) = \prod_{i=1}^{n+1} (x - x_i)\right)$$

does not change sign while $-1 < x_1 < \cdots < x_{n+1} < 1$. If in particular $f \in C^n$, $f^{(n)}$ does not change sign at $(-1, 1)$ and is not identically zero at some nondegenerate interval then Corollary 1 implies that $\mathscr{P}$ is Gatoux differentiable at f.

We turn now to application for spline functions. Let $U_n = S_{m,r}$ be the space of splines of degree $m-1$ with r fixed knots $-1 < \tilde{x}_1 < \cdots < \tilde{x}_r < 1$,

$\dim S_{m,r} = m + r = n$ ($m \geqslant 2$, $r \geqslant 1$). It is well known that $S_{m,r}$ is a weak Chebyshev space [9]. Moreover, by a result of Galkin [1] and Strauss [11] $S_{m,r}$ is also a unicity subspace of C_1. Hence $S_{m,r}$ is a UW-space. Evidently, $S_{m,r}$ fulfils the smoothness and nondegeneracy conditions of Theorem 1. The canonical set of $S_{m,r}$ is given by zeros of a certain perfect spline, see [7].

COROLLARY 2. *Let* $U_n = S_{m,r}$ ($m \geqslant 2$, $r \geqslant 1$). *Then for any* $f \in K(S_{m,r}) \cap \mathrm{Lip}\,1$ *such that* $f - \mathscr{L}_\tau(f)$ *is not identically zero on a nondegenerate interval, the operator* $\mathscr{P}$ *of best* L_1*-approximation is Gatoux differentiable at* f *and* $D_f\mathscr{P} = \mathscr{L}_\tau$.

Corollary 2 holds in particular for functions $f \in C^m$ such that $f^{(m)}$ changes its sign only at $\tilde{x}_1, \ldots, \tilde{x}_r$ and does not vanish on a nondegenerate interval (see [7, p. 8]).

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